Discrete Mathematics Recitation Course Lecture 4

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4-1

Mathematical Induction

4-1 Ex.10

- a) Find a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$

  by examining the values of this expression for small values of n.

- b) Prove the formula you conjectured in part (a).

4-1 Ex.10 (cont' d)

- By computing the first few sums and getting the answers 1/2, 2/3, and 3/4, we guess that the sum is $n/(n+1)$.

- *Basis step:*

  $n = 1$: $n/(n+1) = 1/2$.

- *Induction step:*

  Suppose that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$.

  Then $\left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)}$

  $= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2}$.

4-1 Ex.25

- Prove that if $h > -1$, then $1 + nh \leq (1 + h\,)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality.**

- Let $P(n)$ be "$1 + nh \leq (1 + h\,)^n, h > -1$".
- *Basis step:*
  $P(0)$ is true because $1 + 0 \cdot h = 1 \leq (1 + h\,)^0$.
- *Induction step:*
  Assume $1 + kh \leq (1 + h\,)^k$.
  Then because $(1 + h) > 0$,

$$(1 + h\,)^{k+1} = (1 + h)(1 + h\,)^k \geq (1 + h)(1 + kh)$$
$$= 1 + (k + 1)h + kh^2 \geq 1 + (k + 1)h.$$

4-1 Ex.37

- Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.

- *Basis step:* $11^{1+1} + 12^{2\cdot 1-1} = 121 + 12 = 133$.
- *Inductive step:*

  Assume the inductive hypothesis, that $11^{n+1} + 12^{2n-1}$ is divisible by 133.

  Then $11^{(n+1)+1} + 12^{2(n+1)-1}$

  $$= 11 \cdot 11^{n+1} + 144 \cdot 12^{2n-1}$$

  $$= 11 \cdot 11^{n+1} + (11 + 133) \cdot 12^{2n-1}$$

  $$= 11(11^{n+1} + 12^{2n-1}) + 133 \cdot 12^{2n-1}.$$

4-1 Ex.63

- Show that if $A_1,\ A_2, \ldots,\ A_n$ are sets where $n \geq 2$, and for all pairs of integers i and j with $1 \leq i < j \leq n$ either:

  A_i is a subset of A_j or A_j is a subset of A_i,

  then there is an integer i, $1 \leq i \leq n$ such that A_i is a subset of A_j for all integers j with $1 \leq j \leq n$.

- *Basis step:*

  If $A_1 \subseteq A_2$, then A_1 satisfies the condition of being a subset of each set in the collection; otherwise $A_2 \subseteq A_1$, so A_2 satisfies the condition.

4-1 Ex.63 (cont'd)

- *Inductive step:*

  Assume the inductive hypothesis, that the conditional statement is true for k sets, and suppose we are given $k+1$ sets that satisfy the given conditions.
- By the inductive hypothesis, there must be a set A_i for some $i \le k$ such that $A_i \subseteq A_j$ for $1 \le j \le k$.
- If $A_i \subseteq A_{k+1}$, then we are done.
- Otherwise, we know that $A_{k+1} \subseteq A_i$, and this tells us that A_{k+1} satisfies the condition of being a subset of A_j for $1 \le j \le k+1$.

4-2

Strong Induction and Well-Ordering

4-2 Ex.9

- Use strong induction to prove that $\sqrt{2}$ is irrational. [*Hint:* Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b]

- *Basis step:* $P(1)$ is true because $\sqrt{2} > 1 \geq 1/b$ for all positive integers b.
- *Inductive step:*

  Assume that $P(j)$ is true for all $j \leq k$, where k is an arbitrary positive integer; we prove that $P(k+1)$ is true by contradiction.

4-2 Ex.9 (cont’ d)

- Assume that $\sqrt{2} = \frac{k+1}{b}$ for some positive integer b.
- Then $2b^2 = (k+1)^2$, so $(k+1)^2$ is even, and hence, $k+1$ is even.
- So write $k+1 = 2t$ for some positive integer t, hence $2b^2 = 4t^2$ and $b^2 = 2t^2$.
- By the same reasoning as before, b is even, so $b = 2s$ for some positive integer s.
- Then $\sqrt{2} = \frac{k+1}{b} = \frac{2t}{2s} = t/s$. But $t \leq k$, so this contradicts the inductive hypothesis, and our proof of inductive step is complete.

4-2 Ex.30

- Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

- *Basis step:* $a^0 = 1$ is true by the definition of a^0.
- *Inductive step:* Assume that $a^j = 1$ for all nonnegative integers j with $j \leq k$.

  Then note that $a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1$.

4-2 Ex.30 (cont'd)

- The flaw comes in the inductive step, where we implicitly assuming that $k \geq 1$ in order to talk about a^{k-1} in the denominator.
- Our basis step was $n = 0$, so we are not justified in assuming that $k \geq 1$ when we try to prove the statement for $k+1$ in the inductive step.
- Indeed, it is precisely at $n = 1$ that the proposition breaks down.

4-3

Recursive Definitions
and Structural Induction

4-3 Ex.24

- Give a recursive definition of
  - a) the set of odd positive integers.
  - b) the set of positive integer powers of 3.
  - c) the set of polynomials with integer coefficient.

- a) $1 \in S$; and if $n \in S$, then $n + 2 \in S$.
- b) $3 \in S$; and if $n \in S$, then $3n \in S$.
- c) Assume that the variable for these polynomials is the letter x, and all integers are in S; if $p(x) \in S$ and n is any integer, then $xp(x) + n$ is in S.

4-3 Ex.32

- a) Give a recursive definition of the function *ones*(s), which counts the number of ones in a bit string s.
- b) Use structural induction to prove that *ones*(st) = *ones*(s) + *ones*(t).

- *ones*(λ) = 0 and *ones*(wx) = x + *ones*(w), where w is a bit string and x is a bit
- *Basis step:* when $t = \lambda$, *ones*($s\lambda$) = *ones*(s) + 0 = *ones*(s) + *ones*(λ).
- *Inductive step:*

  write $t = wx$, then *ones*($s(wx)$) = *ones*($(sw)x$) = x + *ones*(sw)

  ≡ x + *ones*(s) + *ones*(w)

  ≡ *ones*(s) + (x + *ones*(w))

  ≡ *ones*(s) + *ones*(wx).

4-3 Ex.40

- Recursively define the set of bit strings that have more zeros than ones.

- 0 is in the set.
- If x and y are in the set, then so are xy, $1xy$, $x1y$, and $xy1$.

4-4

Recursive Algorithms

4-4 Ex.17

- Describe a recursive algorithm for multiplying two nonnegative integers x and y based on the fact that $xy = 2(x \cdot (y/2))$ when y is even and $xy = 2(x \cdot \lfloor y/2 \rfloor) + x$ when y is odd, together with the initial condition $xy = 0$ when $y = 0$.

- **Procedure** *multiply*$(x, y$: nonnegative integers)

  if $y = 0$ **then** *multiply*$(x, y) := 0$

  else if y is even **then**

  multiply$(x, y) := 2 \cdot$ *multiply*$(x, y/2)$

  else *multiply*$(x, y) := 2 \cdot$ *multiply*$(x, (y-1)/2) + x$

4-4 Ex.23

- Devise a recursive algorithm for computing n^2 where n is a nonnegative integer using the fact that $(n+1)^2 = n^2 + 2n + 1$. Then prove that this algorithm is correct.

- **procedure** *square*$(n$: nonnegative integer)
  if *n* = 0 **then** *square*$(n) := 0$
  else *square*$(n) :=$ *square*$(n-1) + 2(n-1) + 1$

4-4 Ex.23 (cont'd)

- Let $P(n)$ be the statement that this algorithm correctly computes n^2.
- Because $0^2 = 0$, then algorithm works correctly if the input is 0.
- Assume that the algorithm works correctly for input k. Then for input $k + 1$, it gives as output its output when the input is k, plus $2(k + 1 - 1) + 1$.
- By the inductive hypothesis, its output at k is k^2, so its output at $k + 1$ is $k^2 + 2(k + 1 - 1) + 1 = k^2 + 2k + 1 = (k + 1)^2$, as desired.

4-4 Ex.29

- Devise a recursive algorithm to find the nth term of the sequence defined by $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1} \cdot a_{n-2}$, for $n = 2, 3, 4 \ldots$

- **procedure** $a(n$: nonnegative integer)

  if $n = 0$ **then** $a(n) := 1$

  else if $n = 1$ **then** $a(n) := 2$

  else $a(n) := a(n - 1) * a(n - 2)$

4-4 Ex.38

- Give a recursive algorithm for finding the string w^i, the concatenation of i copies of w, when w is a bit string.

- **procedure** *power*$(w$: bit string, i: nonnegative integer)
  if $i = 0$ **then** *power*$(w, i) := \lambda$
  else *power*$(w, i) := w$ concatenation with *power*$(w, i - 1)$